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Leaky cavities with unwanted noise

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A phenomenological approach is developed that allows one to completely describe the effects of unwanted noise, such as the noise associated with absorption and scattering, in high-Q cavities. This noise is modeled by a block of beam splitters and an additional input-output port. The replacement schemes enable us to formulate appropriate quantum Langevin equations and input-output relations. It is demonstrated that unwanted noise renders it possible to combine a cavity input mode and the intracavity mode in a nonmonochromatic output mode. Possible applications to unbalanced and cascaded homodyning of the intracavity mode are discussed and the advantages of the latter method are shown.

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I. INTRODUCTION

Cavity quantum electrodynamics (cavity QED) has been a powerful tool in a lot of investigations dealing with fundamentals of quantum physics and applications such as quantum information processing, for a review see, e. g., Refs. [1, 2]. It has offered a number of proposals for quantum-state generation, manipulation, and transfer between remote nodes in quantum networks. A cavity is a resonatorlike device with one or more fractionally transparent mirrors characterized by small transmission coefficients such that large quality values Q can be realized. Hence one may regard the mode spectrum of the intracavity field as consisting of narrow lines. As a rule, excited atoms inside the cavity serve as source of radiation, and the fractionally transparent mirrors are used to release radiation for further applications and to feed radiation in the cavity in order to modify the intracavity field and thereby the outgoing field either.

Manipulations with atoms in cavities and cavity fields give a number of possibilities of quantum-state engineering, see, e. g., Refs. [3, 4]. For example, schemes for the generation of arbitrary field states have been proposed [5]. Further, proposals for the generation of entangled-states of light have been made (see, e.g., Ref. [6]). It is worth noting that, using the technique of adiabatic atom transitions coherent superposition states of the radiation field inside the cavity can be prepared [7].

The field escaping from an excited cavity has been proposed to be used for homodyne detection of the intracavity mode and reconstruction of its quantum state [8]. Since the output mode is a nonmonochromatic one, this proposal is based on an operational definition of the Wigner function. The employment of cavities as remote nodes in quantum networks has been proposed [9]. Laser driving of atoms allows one to create such a specific pulse of the output mode which is completely coupled into another cavity. This can be used for transferring quantum states between spatially separated atoms trapped inside cavities. Cavities are also important in optical parametric amplification frequently used for the generation of squeezed states [10].

One of the most crucial points in implementing the proposed schemes such as the ones mentioned above is the decoherence. It appears due to the uncontrolled interaction of the radiation with some external degrees of freedom giving rise to absorption and scattering of the radiation one is interested in. In this context, a serious drawback is the fact that for high-Q optical cavities, at least with the presently available technology, such unwanted losses can be of the same order of magnitude as the wanted losses associated with the transmittance of the coupling mirrors [11, 12, 13]. Thus, nonclassical features of the outgoing field can be substantially reduced compared with the corresponding properties of the intracavity field [14].

There exist several approaches to the theoretical description of leaky cavities for the idealized case that unwanted losses can be ignored. Within the framework of quantum noise theory (QNT), in Ref. [15] each intracavity mode is linearly coupled, through one or more fractionally transparent mirrors, with a continuum of external modes forming dissipative systems for the intracavity modes. Based on Markovian approximation, it can be concluded that the intracavity modes obey quantum Langevin equations. The external field is composed of two kinds of fields: input and output ones, where the input field gives rise to the Langevin noise forces. Moreover, the input and output fields are related to each other by means of input-output relation.

The quantum field theoretical (QFT) approaches to the problem are based on (macroscopic) QED. So Refs. [16, 17, 18] start directly from an ordinary continuous-mode expansion of the electromagnetic field in the presence of passive, nonabsorbing media [19, 20]. Under certain conditions, this approach also leads to a description of the cavity in terms of quantum Langevin equations and input-output relation. In another version

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of the QFT approach [21], solutions of Maxwell's equations are constructed by using Feshbach's projection formalism [22]. Separating from the beginning all degrees of freedom into two parts—internal and external ones—one can also obtain, in some approximation, the Hamiltonian used in QNT.

As already mentioned, the standard versions of both the QNT approach and the QFT approach do not take into account the presence of unwanted losses and hence, the additional, unwanted noise unavoidably associated with them. Thus, these theories cannot be applied to realistic situations, in general. Within an extended version of the QNT approach, the unwanted losses the intracavity field suffers from can be modeled by introduction into the Langevin equations additional damping and noise terms, which corresponds to the introduction into the system of additional input-output ports [14, 21]. The applicability of this model is restricted, in general, to the case of all the input ports being unused.

More recently a QFT approach to the description of a leaky cavity with unwanted noise has been presented [23]. Applying quantization of the electromagnetic field in dispersing and absorbing media [20, 25], a generalized Langevin equation and input-output relation have been derived. An extended version of the QNT approach can be obtained by applying the model of imperfect coupling between two systems, see, e. g., Ref. [26]. In this scheme, a unidirectional coupling of the considered systems is studied, the unwanted noise being modeled by beam splitters inserted in the input and output channels. For transferring such a method to the description of a cavity with unwanted noise, one must carefully check the completeness of the parametrization of the considered replacement schemes.

In the present paper we generalize, by means of replacement schemes, the QNT approach with the aim to complete both the quantum Langevin equations and the input-output relation in a consistent way, such that unwanted noise is fully included in the theory—in full agreement with the QFT approach in Ref. [23]. The analysis will show that there exist different formulations of the theory. Favoring one over the other may depend on the physical conditions and on the available information on the cavity. Moreover, we will demonstrate that an incomplete description of the unwanted noise may ignore important physical effects. As an example it is shown that the unwanted noise may lead to the combination of a cavity input mode with the intracavity mode in a non-monochromatic output mode. Such mode matching does not occur in an ideal leaky cavity or in some incomplete models of unwanted noise effects.

The paper is organized as follows. In Sec. II a beam-splitter-based replacement scheme is introduced which is suitable for modeling the unwanted noise of a one-sided cavity. Both the quantum Langevin equation and the input-output relation associated with the replacement scheme are presented. The relations between the c-number coefficients in these equations are derived. Section III is devoted to the problem of consistency and completeness of a given quantum Langevin equation together with the corresponding input-output relations. It is shown that the requirement of preserving commutation rules necessarily leads to constraints on the c-number coefficients in the theory. The effect of noise-induced mode coupling between intracavity and input modes is considered in Sec. IV. Section V is devoted to the application of this effect to the problem of unbalanced and cascaded homodyning of the intracavity mode. Finally, a summary and some concluding remarks are given in Sec. VI.

II. UNWANTED NOISE

As mentioned in the Introduction, the unwanted loss of cavity photons due to scattering and absorption can be modeled by appropriately chosen input and output ports. This is sketched in Fig. 1 in the simplest case for a one-sided cavity, where the operators $\hat{d}_{\rm in}(t)$ and $\hat{d}_{\rm out}(t)$, respectively, correspond to the wanted radiative input and output, whereas the operators $\hat{c}_{\rm in}(t)$ and $\hat{c}_{\rm out}(t)$, respectively, correspond to input and output channels associated with unwanted noise.

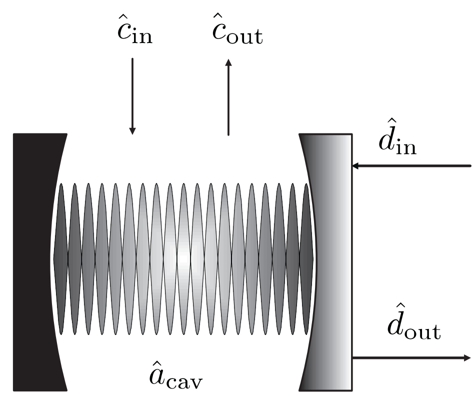

FIG. 1: One-sided cavity with unwanted internal losses.

The scheme is formally equivalent to a four-port cavity having two fractionally transparent mirrors as considered in Ref. [15]. Hence any (single-mode) cavity operator $\hat{a}_{\rm cav}$ can be assumed to obey a quantum Langevin equation of the type

$$\begin{aligned}\dot{\hat{a}}_{\rm cav}(t) = &-\left[i\omega_0 + \tfrac{1}{2}\left(\gamma + |\mathcal{A}|^2\right)\right]\hat{a}_{\rm cav}(t)\\ &+ \sqrt{\gamma}\,\hat{d}_{\rm in}\left(t\right) + \mathcal{A}\hat{c}_{\rm in}\left(t\right),\end{aligned} \tag{1}$$

and the corresponding input-output relation reads as

$$\hat{d}_{\rm out}\left(t\right) = \sqrt{\gamma}\,\hat{a}_{\rm cav}\left(t\right) - \hat{d}_{\rm in}\left(t\right). \tag{2}$$

Here, ω_0 is the resonance frequency of the cavity, γ is the decay rate caused by the wanted output channel, and $|\mathcal{A}|^2$ is the part of the decay rate due to unwanted internal noise, where, for some reason which will be clarified

later, $\mathcal{A}$ is assumed to be a complex number. Note that such an approach has effectively been used in Ref. [14] for analyzing the quantum-state extraction from a cavity in the presence of unwanted losses. It is useful when the input field is in the vacuum state. In this case the possibility of absorption or scattering of input photons plays no role.

A. Noisy coupling mirror

In the most general case one may use the input port of a cavity for different purposes, e.g., for combining the intracavity and input modes in an output mode. This possibility can be useful for the quantum-state reconstruction as considered in Sec. V. For a correct description of such a process one should take into account that input photons can be absorbed or scattered by the coupling mirror before entering into the cavity.

The corresponding type of unwanted noise can be included in the theory in a systematic way by applying the concept of replacement schemes as follows. Instead of considering the actual coupling mirror, we consider an ideal semitransparent mirror that does not give rise to unwanted noise and we model the unwanted noise by inserting appropriately chosen beam splitters in the input and output channels of the cavity, as sketched in Fig. 2. Clearly, the symmetric beam splitters BS_1 and BS_2, respectively, are closely related to the unwanted losses that the input and output fields suffer when passing through the coupling mirror. Moreover it will turn out that a third beam splitter BS_3 is required, which simulates some feedback. This (asymmetric) beam splitter is allowed to realize an U(2)-group transformation, thereby introducing an additional phase shift (see Appendix A and Refs. [20, 27]). Including such a feedback into the theory ensures one to describe all kinds of unwanted noise in high-Q cavities. The corresponding proof is given in Sec. III.

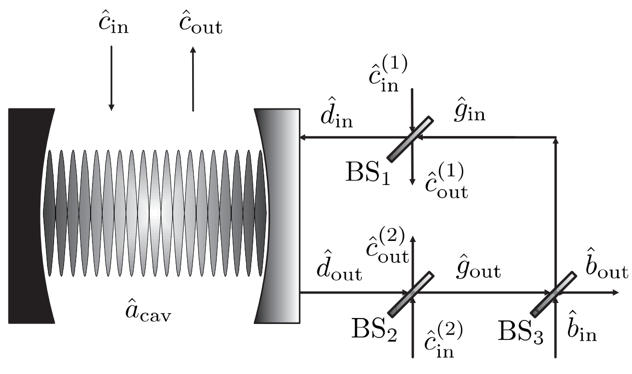

FIG. 2: Replacement scheme for modeling the unwanted noise in a one-sided cavity. The symmetrical SU(2)-type beam splitters BS_1 and BS_2 model the unwanted noise in the coupling mirror, and the asymmetrical U(2)-type beam splitter BS_3 simulates some feedback.

Using Eqs. (1) and (2) and the input-output relations for each beam-splitter (Appendix A), we obtain the extended quantum Langevin equation

$$\dot{\hat{a}}_{\rm cav}(t) = -\left[i\omega_{\rm cav} + \tfrac{1}{2}\Gamma\right]\hat{a}_{\rm cav}(t) + \mathcal{T}^{(\rm c)}\hat{b}_{\rm in}(t) + \mathcal{A}^{(\rm c)}_{(1)}\hat{c}^{(1)}_{\rm in}(t) + \mathcal{A}^{(\rm c)}_{(2)}\hat{c}^{(2)}_{\rm in}(t) + \mathcal{A}\hat{c}_{\rm in}(t) \tag{3}$$

and the extended input-output relation

$$\hat{b}_{\rm out}(t) = \mathcal{T}^{(\rm o)}\hat{a}_{\rm cav}(t) + \mathcal{R}^{(\rm o)}\hat{b}_{\rm in}(t) + \mathcal{A}^{(\rm o)}_{(1)}\hat{c}^{(1)}_{\rm in}(t) + \mathcal{A}^{(\rm o)}_{(2)}\hat{c}^{(2)}_{\rm in}(t) \tag{4}$$

for a cavity in the presence of unwanted noise (Appendix B). Here,

$$\Gamma = \gamma\,\frac{1-\left|\mathcal{R}^{(3)}\right|^2\left|\mathcal{T}^{(1)}\right|^2\left|\mathcal{T}^{(2)}\right|^2}{\left|1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}\right|^2} + |\mathcal{A}|^2 \tag{5}$$

is the cavity decay rate and

$$\omega_{\rm cav} = \omega_0 - i\frac{\gamma}{2}\,\frac{\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)} - \mathcal{R}^{(3)}\mathcal{T}^{(1)*}\mathcal{T}^{(2)*}}{\left|1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}\right|^2} \tag{6}$$

is the shifted frequency of the cavity mode. The other c-number coefficients are defined as follows:

$$\mathcal{T}^{(\rm c)} = \sqrt{\gamma}\,\frac{\mathcal{T}^{(1)}\mathcal{T}^{(3)*}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{7}$$

$$\mathcal{A}^{(\rm c)}_{(1)} = \sqrt{\gamma}\,\frac{\mathcal{R}^{(1)}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{8}$$

$$\mathcal{A}^{(\rm c)}_{(2)} = -\sqrt{\gamma}\,\frac{\mathcal{T}^{(1)}\mathcal{R}^{(2)}\mathcal{R}^{(3)*}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{9}$$

$$\mathcal{T}^{(\rm o)} = \sqrt{\gamma}\,e^{i\varphi^{(3)}}\,\frac{\mathcal{T}^{(2)}\mathcal{T}^{(3)}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{10}$$

$$\mathcal{R}^{(\rm o)} = e^{i\varphi^{(3)}}\,\frac{\mathcal{R}^{(3)}-\mathcal{T}^{(1)}\mathcal{T}^{(2)}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{11}$$

$$\mathcal{A}^{(\rm o)}_{(1)} = -e^{i\varphi^{(3)}}\,\frac{\mathcal{T}^{(2)}\mathcal{R}^{(1)}\mathcal{T}^{(3)}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{12}$$

$$\mathcal{A}^{(\rm o)}_{(2)} = e^{i\varphi^{(3)}}\,\frac{\mathcal{R}^{(2)}\mathcal{T}^{(3)}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,, \tag{13}$$

where $\mathcal{T}^{(k)}$ and $\mathcal{R}^{(k)}$, respectively, are the transmission and reflection coefficients of the kth beam splitter, and $\varphi^{(3)}$ is a phase factor attributed to the third beam splitter.

We see that the replacement scheme in Fig. 2 leads to a description of the cavity in terms of the quantum Langevin equation (3) and input-output relation (4) which are suited to include unwanted noise in the theory. The corresponding coefficients are expressed via the parameters of the component parts of the replacement scheme—the cavity (with a coupling mirror that is free of unwanted losses) and three beam splitters. It is worth noting that the results obtained are in agreement with those derived on the basis of the QFT approach in Ref. [23].

B. Commutation relations

Clearly, the c-number coefficients in Eqs. (3) and (4) are not independent of each other, since they ensure, by construction, the validity of the commutation relations

$$[\hat{a}_{\mathrm{cav}}(t),\hat{a}^{\dagger}_{\mathrm{cav}}(t)]=1, \tag{14}$$

$$\left[\hat{b}_{\mathrm{out}}(t_1),\hat{b}^{\dagger}_{\mathrm{out}}(t_2)\right]=\delta(t_1-t_2) \tag{15}$$

Vice versa, if the commutation relations (14) and (15) are assumed to be valid, then from the quantum Langevin equation (3) together with the input-output relation (4) and the commutation relations

$$\left[\hat{b}_{\mathrm{in}}(t_1),\hat{b}^{\dagger}_{\mathrm{in}}(t_2)\right]=\delta(t_1-t_2), \tag{16}$$

$$\left[\hat{c}_{\mathrm{in}}(t_1),\hat{c}^{\dagger}_{\mathrm{in}}(t_2)\right]=\delta(t_1-t_2), \tag{17}$$

$$\left[\hat{c}^{(1)}_{\mathrm{in}}(t_1),\hat{c}^{(1)\dagger}_{\mathrm{in}}(t_2)\right]=\delta(t_1-t_2), \tag{18}$$

$$\left[\hat{c}^{(2)}_{\mathrm{in}}(t_1),\hat{c}^{(2)\dagger}_{\mathrm{in}}(t_2)\right]=\delta(t_1-t_2) \tag{19}$$

it necessarily follows that relations between the mentioned coefficients must exist. Note that mixed commutators vanish as a natural consequence of the assumption that the cavity mode, the external modes, and the dissipative systems responsible for the unwanted noise are assumed to refer to different degrees of freedom.

Inserting the solution of the quantum Langevin equation (3)

$$\begin{aligned}\hat{a}_{\mathrm{cav}}(t)&=\hat{a}_{\mathrm{cav}}(0)e^{-(i\omega_{\mathrm{cav}}+\Gamma/2)t}\\&+\int_0^t \mathrm{d}t' e^{-(i\omega_{\mathrm{cav}}+\Gamma/2)(t-t')}\big[\mathcal{T}^{(\mathrm{c})}\hat{b}_{\mathrm{in}}(t')\\&\quad+\mathcal{A}^{(\mathrm{c})}_{(1)}\hat{c}^{(1)}_{\mathrm{in}}(t')+\mathcal{A}^{(\mathrm{c})}_{(2)}\hat{c}^{(2)}_{\mathrm{in}}(t')+\mathcal{A}\hat{c}_{\mathrm{in}}(t')\big]\end{aligned} \tag{20}$$

in the left-hand side of Eq. (14), assuming that

$$\left[\hat{a}_{\mathrm{cav}}(0),\hat{a}^{\dagger}_{\mathrm{cav}}(0)\right]=1, \tag{21}$$

and taking into account Eqs. (16–19), we find that Eq. (14) holds true only if the condition

$$\Gamma=\left|\mathcal{A}\right|^2+\left|\mathcal{A}^{(\mathrm{c})}_{(1)}\right|^2+\left|\mathcal{A}^{(\mathrm{c})}_{(2)}\right|^2+\left|\mathcal{T}^{(\mathrm{c})}\right|^2 \tag{22}$$

is satisfied. Similarly, inserting Eq. (4), together with $\hat{a}_{\mathrm{cav}}(t)$ from Eq. (20), in the left-hand side of Eq. (15), we can easily see that Eq. (15) holds true if the conditions

$$\left|\mathcal{R}^{(\mathrm{o})}\right|^2+\left|\mathcal{A}^{(\mathrm{o})}_{(1)}\right|^2+\left|\mathcal{A}^{(\mathrm{o})}_{(2)}\right|^2=1 \tag{23}$$

and

$$\mathcal{T}^{(\mathrm{o})}+\mathcal{T}^{(\mathrm{c})*}\mathcal{R}^{(\mathrm{o})}+\mathcal{A}^{(\mathrm{c})*}_{(1)}\mathcal{A}^{(\mathrm{o})}_{(1)}+\mathcal{A}^{(\mathrm{c})*}_{(2)}\mathcal{A}^{(\mathrm{o})}_{(2)}=0 \tag{24}$$

are satisfied. Needless to say that substituting Eqs. (5) and (7)–(13) into Eqs. (22)–(24) and utilizing Eq. (A5) yields identities.

III. CONSISTENCY AND COMPLETENESS

There exist some other approaches to the problem of unwanted noise in cavities, which may lead to quantum Langevin equations and input-output relations different from Eqs. (3) and (4); see, e.g., Ref. [21]. Hence the question of equivalence and completeness of different types of quantum Langevin equations and the input-output relations associated with them arises. Answering the question is no trivial task, and, in fact, some approaches describe cavities which do not describe all the typical situations.

Quite general, the quantum Langevin equation and the input-output relation can be written in the form

$$\dot{\hat{a}}_{\mathrm{cav}}=-\left[i\omega_{\mathrm{cav}}+\tfrac{1}{2}\Gamma\right]\hat{a}_{\mathrm{cav}}+\mathcal{T}^{(c)}\hat{b}_{\mathrm{in}}(t)+\hat{C}^{(c)}(t), \tag{25}$$

$$\hat{b}_{\mathrm{out}}(t)=\mathcal{T}^{(\mathrm{o})}\hat{a}_{\mathrm{cav}}(t)+\mathcal{R}^{(\mathrm{o})}\hat{b}_{\mathrm{in}}(t)+\hat{C}^{(o)}(t). \tag{26}$$

where the operators $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$ should obey the commutation relations

$$\left[\hat{C}^{(c)}(t_1),\hat{C}^{(c)\dagger}(t_2)\right]=\left|\mathcal{A}^{(c)}\right|^2\delta(t_1-t_2), \tag{27}$$

$$\left[\hat{C}^{(o)}(t_1),\hat{C}^{(o)\dagger}(t_2)\right]=\left|\mathcal{A}^{(o)}\right|^2\delta(t_1-t_2), \tag{28}$$

$$\left[\hat{C}^{(c)}(t_1),\hat{C}^{(o)\dagger}(t_2)\right]=\left|\mathcal{A}^{(c)}\right|\left|\mathcal{A}^{(o)}\right|e^{i\kappa}\cos\zeta\,\delta(t_1-t_2). \tag{29}$$

Here $\left|\mathcal{A}^{(c)}\right|$, $\left|\mathcal{A}^{(o)}\right|$, and $e^{i\kappa}\cos\zeta$ are coefficients that along with $\mathcal{T}^{(c)}$, $\mathcal{T}^{(o)}$, $\mathcal{R}^{(o)}$, and Γ satisfy the constraints

$$\Gamma=\left|\mathcal{A}^{(\mathrm{c})}\right|^2+\left|\mathcal{T}^{(\mathrm{c})}\right|^2, \tag{30}$$

$$\left|\mathcal{R}^{(\mathrm{o})}\right|^2+\left|\mathcal{A}^{(\mathrm{o})}\right|^2=1, \tag{31}$$

$$\mathcal{T}^{(\mathrm{o})}+\mathcal{T}^{(\mathrm{c})*}\mathcal{R}^{(\mathrm{o})}+\left|\mathcal{A}^{(c)}\right|\left|\mathcal{A}^{(o)}\right|e^{i\kappa}\cos\zeta=0, \tag{32}$$

which follow, in a similar way as outlined for the scheme in Sec. II B, from the requirement of preserving the commutation rules.

The constraints (30)–(32) [or (22)–(24) in the case of the scheme in Fig. 2] mean that the c-number coefficients in Eqs. (25-29) cannot be chosen freely, but can take values only on a certain manifold. In this context, Eqs. (5)–(13) can be considered as an example of a parametrization of this manifold, where the number of parameters describing the component part of the replacement scheme in Fig. 2 exactly coincides with the dimensionality of the manifold.

However, one may also think of parameterizations that do not cover the whole manifold. In this case the corresponding replacement scheme—referred to as a degenerate scheme—does not describe all possible cavities. In

order to test as to whether a given parametrization is associated with a degenerate scheme, one can apply an appropriate theorem of differential geometry [28]. For this purpose one should first present Eqs. (5)–(13) in the form of real functions of real arguments. Next, one should find the rank of the matrix constructed from the first derivatives of these functions and compare it with the dimensionality of the manifold. For the replacement scheme in Fig. 2 this has been checked using MATHEMATICA. As expected, it has turned out that the scheme is nondegenerate. Hence the scheme leads to a complete and consistent description of a (one-sided) cavity with unwanted noise.

Another possibility to express the c-number coefficients in terms of independent parameters follows from Eqs. (30-32). One can simply consider the coefficients $\mathcal{T}^{(c)}$, $\mathcal{T}^{(o)}$, $\mathcal{R}^{(o)}$, Γ and $\omega_{\rm cav}$ as independent parameters. The coefficients $\left|\mathcal{A}^{(c)}\right|$, $\left|\mathcal{A}^{(o)}\right|$ and $e^{i\kappa}\cos\zeta$, describing the unwanted noise, can be expressed in terms of them. The values of such independent parameters have to belong to the manifold defined by Eqs. (30-32).

It is worth noting that the operators $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$ in Eqs. (25) and (26) can be expanded as

$$\hat{C}^{(c)}(t) = \sum_k \mathcal{A}^{(\rm c)}_{(k)} \hat{c}^{(k)}_{\rm in}(t), \tag{33}$$

$$\hat{C}^{(o)}(t) = \sum_k \mathcal{A}^{(\rm o)}_{(k)} \hat{c}^{(k)}_{\rm in}(t). \tag{34}$$

This implies that different representations (and corresponding replacement schemes) of the operators of the unwanted noise can be obtained. The quantum Langevin equation and the input-output relation in the form of Eqs. (3) and (4) are an example of such a representation, where the operators of the unwanted noise are expanded in a three-dimensional space. However, it is clear that two operators $\hat{C}^{(c)}(t_1)$ and $\hat{C}^{(o)}(t_1)$ can be expanded in two-dimensional basis. Hence, for the complete characterization of a cavity, one can use only two independent sources of unwanted noise.

As already mentioned, there exist schemes which do not describe all the physically possible lossy cavities and, in fact, describe special cases of lossy cavities. In other words, the corresponding parameters do not cover the whole manifold of the values of the coefficients in Eqs. (25) and (26) which are, in principle, possible and hence, the schemes can be considered as being degenerate. An example of such scheme is the replacement scheme in Fig. 3. The corresponding quantum Langevin equation and the input-output relation, which are special cases of Eqs. (25) and (26), read

$$\dot{\hat{a}}_{\rm cav} = -\left[i\omega_{\rm cav} + \tfrac{1}{2}\Gamma\right]\hat{a}_{\rm cav} + \mathcal{T}^{(c)}\hat{b}_{\rm in}(t) + \mathcal{A}^{(\rm c)}_{(1)}\hat{c}^{(1)}_{\rm in}(t), \tag{35}$$

$$\hat{b}_{\rm out}(t) = \mathcal{T}^{(o)}\hat{a}_{\rm cav}(t) + \mathcal{R}^{(o)}\hat{b}_{\rm in}(t) + \mathcal{A}^{(\rm o)}_{(1)}\hat{c}^{(1)}_{\rm in}(t) + \mathcal{A}^{(\rm o)}_{(2)}\hat{c}^{(2)}_{\rm in}(t) \tag{36}$$

The parametrization can easily be obtained from the parametrization (5)–(13) by setting therein $\mathcal{T}^{(3)}=1$, $\mathcal{R}^{(3)}=0$ and $\mathcal{A}=0$.

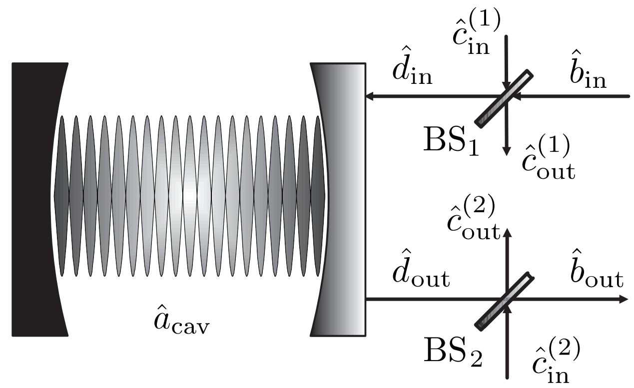

FIG. 3: An example of a degenerate replacement scheme.

It is not difficult to prove that for this scheme, along with Eqs. (30)–(32), the additional constraint

$$\frac{\mathcal{T}^{(\rm o)}\mathcal{T}^{(\rm c)}}{\Gamma} + \mathcal{R}^{(\rm o)} = 0 \tag{37}$$

is satisfied. This relation does not follow from the requirement of preserving the commutation rules (14) and (15). Clearly, the rank of the matrix constructed from the first derivatives of the real functions corresponding to the parametrization is not equal to the number of the independent coefficients in Eqs. (35) and (36)—a sign that the scheme is indeed degenerate.

It is worth noting that the physics behind this degenerate scheme is closely related to that of a cavity without unwanted noise. This becomes clear from the following argument. The loss channels modeled by the two beam splitters may equivalently be interpreted as the losses that the input (output) field suffers from before entering (after leaving) the cavity. A consequence of this fact is that the losses modeled in this way cannot affect the decay rate of the intracavity mode. Thus the unwanted losses do not affect the dynamics of the intracavity mode.

IV. NOISE-INDUCED MODE COUPLING

In the generation and processing of nonclassical radiation one is commonly interested in a reduction of unwanted noise, because it gives rise to quantum decoherence, in general. However, if the input port of a cavity is used, the presence of unwanted losses does not only change the properties of the intracavity mode and the outgoing field. In this case, a new possibility for combining the intracavity mode and an input mode in a nonmonochromatic output mode appears—a surprising property, which does not exist for cavities without unwanted noise channels. Moreover, such an effect cannot be properly described by a degenerate cavity model such as that given in Fig 3.

After sufficiently long time, $t \gg 1/\Gamma$, the internal field of an ideal cavity (i.e., a cavity without unwanted noise) is completely transferred into the nonmonochromatic cavity-associated output mode (CAOM). Since the efficiency of this process is equal to one, an input signal cannot be reflected into this mode. Therefore, in order to combine the intracavity mode and an input mode in an output mode, one must decrease this efficiency. Realistic cavities are always characterized by some unwanted losses, such as absorption and scattering. Hence, the efficiency of intracavity mode escaping from such a cavity is less than unity [14] and an input mode can be reflected, in principle, into the CAOM. Therefore, combining the intracavity mode with an input mode in the output mode becomes possible.

Assuming that the quantum state of intracavity mode is generated at the zero point of time, the solution of the quantum Langevin equation (25) can be written as

$$\begin{aligned} \hat{a}_{\mathrm{cav}}\left(t_1\right) &= \hat{a}_{\mathrm{cav}}\left(0\right) \mathcal{T}^{(\mathrm{o})-1} F^*\left(t_1\right) \\ &+ \mathcal{T}^{(\mathrm{c})} \mathcal{T}^{(\mathrm{o})-1} \int_0^{t_1} \mathrm{d}t_2\, \xi^*\left(t_1, t_2\right) \hat{b}_{\mathrm{in}}\left(t_2\right) + \hat{\tilde{C}}\left(t_1\right), \end{aligned} \tag{38}$$

where

$$F^*\left(t_1\right) = \mathcal{T}^{(\mathrm{o})} e^{-\left(i\omega_{\mathrm{cav}} + \frac{\Gamma}{2}\right) t_1} \Theta\left(t_1\right), \tag{39}$$

$$\xi^*\left(t_1, t_2\right) = \mathcal{T}^{(\mathrm{o})} e^{-\left(i\omega_{\mathrm{cav}} + \frac{\Gamma}{2}\right)(t_1 - t_2)} \Theta\left(t_1\right) \Theta\left(t_1 - t_2\right), \tag{40}$$

$\Theta\left(t_1\right)$ is a unit step function and $\hat{\tilde{C}}\left(t_1\right)$ is a linear integral expression containing the operators of unwanted noise. Since we assume that the unwanted-noise systems are in the vacuum state, the explicit form of $\hat{\tilde{C}}\left(t_1\right)$ plays no role for our further consideration. Substituting Eq. (38) into the input-output relation (26), one obtains the relation

$$\begin{aligned} \hat{b}_{\mathrm{out}}\left(t_1\right) &= \hat{a}_{\mathrm{cav}}(0) F^*\left(t_1\right) \\ &+ \int_{-\infty}^{+\infty} \mathrm{d}t_2\, G^*\left(t_1, t_2\right) \hat{b}_{\mathrm{in}}\left(t_2\right) + \hat{C}\left(t_1\right). \end{aligned} \tag{41}$$

Hence, the output-mode operator is expressed by the input-mode operator, the intracavity mode operator at the initial time and the operators of unwanted noise. Here

$$G^*\left(t_1, t_2\right) = \mathcal{T}^{(\mathrm{c})} \xi^*\left(t_1, t_2\right) + \mathcal{R}^{(\mathrm{o})} \delta\left(t_1 - t_2\right), \tag{42}$$

and the operator $\hat{C}\left(t_1\right)$ is again a linear integral expression containing the operators of unwanted noise, whose explicit form is not needed for the further considerations. The first term in Eq. (41) describes the extraction of the intracavity mode into the CAOM. The second term describes the reflection of the input field, where $G^*\left(t_1, t_2\right)$ is the integral kernel of the corresponding mode transformation. It is worth noting that non-Hermitian properties of this integral transformation lead to changing (decreasing) the norm of the reflected pulse compared with the input one. This corresponds to the partial absorption/scattering during reflection at the cavity.

For our purposes it is convenient to use another (equivalent) representation of Eq. (41). Let $\left\{U_n^{\mathrm{in}}\left(t_1\right), n = 0, \ldots, +\infty\right\}$ and $\left\{U_n^{\mathrm{out}}\left(t_1\right), n = 0, \ldots, +\infty\right\}$ be two different complete sets of orthogonal functions associated with the input and output modes respectively, i.e.,

$$\hat{b}_{\mathrm{in(out)}}(t_1) = \sum_{n=0}^{+\infty} U_n^{\mathrm{in(out)}}\left(t_1\right) \hat{b}_{\mathrm{in(out);n}}\,, \tag{43}$$

$$\hat{b}_{\mathrm{in(out);n}} = \int_{-\infty}^{+\infty} \mathrm{d}t_1 U_n^{\mathrm{in(out)}*}(t_1)\, \hat{b}_{\mathrm{in(out)}}(t_1)\,. \tag{44}$$

Here $\hat{b}_{\mathrm{in(out)};n}$ is the annihilation operator of an input (output) photon in the nonmonochromatic mode corresponding to the function $U_n^{\mathrm{in(out)}}(t_1)$. We choose the function $U_0^{\mathrm{out}}(t_1)$ in the form of the CAOM,

$$U_0^{\mathrm{out}}(t_1) = \frac{F^*\left(t_1\right)}{\sqrt{\eta_{\mathrm{ext}}}}, \tag{45}$$

where

$$\eta_{\mathrm{ext}} = \int_{-\infty}^{+\infty} \mathrm{d}t_1 \left|F\left(t_1\right)\right|^2 = \frac{\left|\mathcal{T}^{(o)}\right|^2}{\Gamma} \tag{46}$$

can be interpreted as the efficiency of the intracavity-field extraction into the CAOM [14]. The function $U_0^{\mathrm{in}}(t_1)$, defined by using the integral kernel $G\left(t_2, t_1\right)$ as

$$\begin{aligned} U_0^{\mathrm{in}}(t_1) &= \frac{1}{\sqrt{\underline{\eta}_{\mathrm{ref}}}} \int_{-\infty}^{+\infty} \mathrm{d}t_2\, G(t_2, t_1)\, U_0^{\mathrm{out}}(t_2) \\ &= U_0^{\mathrm{out}}(t_1)\, e^{-i\varphi}, \end{aligned} \tag{47}$$

corresponds to the nonmonochromatic matched input mode (MIM), which only makes a contribution, among the other orthogonal input modes of this set, into the CAOM under reflection at the cavity. Here

$$\underline{\eta}_{\mathrm{ref}} = \left|\frac{\mathcal{T}^{(\mathrm{o})} \mathcal{T}^{(\mathrm{c})}}{\Gamma} + \mathcal{R}^{(\mathrm{o})}\right|^2 \tag{48}$$

is the efficiency of the MIM reflection into the CAOM, which can be found through the condition of normalization for the function $U_0^{\mathrm{in}}(t_1)$, Eq. (47). The phase φ is defined as

$$\varphi = \arg\left[\frac{\mathcal{T}^{(\mathrm{o})} \mathcal{T}^{(\mathrm{c})}}{\Gamma} + \mathcal{R}^{(\mathrm{o})}\right]. \tag{49}$$

Along with the CAOM, the MIM is reflected into another nonmonochromatic output mode as well, see Fig. 4. This additional output mode (AOM) results in noise effects when one measures some properties of the quantum

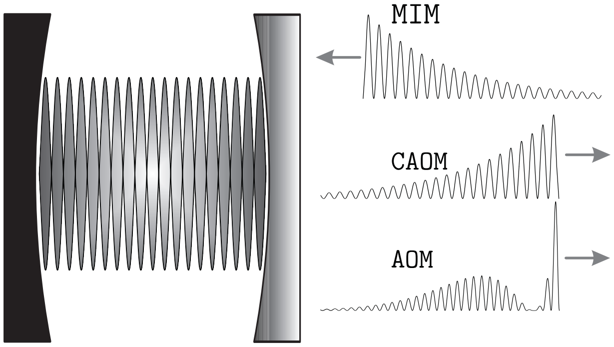

FIG. 4: The mode structure of the external field: cavity-associated output mode (CAOM), additional output mode (AOM), and matched input mode (MIM).

state of the CAOM. To analyze it, we need the total response of the cavity on the MIM, that can be obtained by using the integral kernel $G^{*}\left(t_1,t_2\right)$ as

$$\begin{aligned} U^{\rm out}(t_1)&= \int_{-\infty}^{+\infty} {\rm d}t_2\, G^{*}(t_1,t_2)\, U_0^{\rm in}(t_2) \\ &= \sqrt{\Gamma}\Big(\mathcal{T}^{(\rm c)}\mathcal{T}^{(\rm o)}t_1+\mathcal{R}^{(\rm o)}\Big) \\ &\times e^{-\left(i\omega_{\rm cav}+\frac{\Gamma}{2}\right)t_1+i\left(\arg\mathcal{T}^{(\rm o)}-\varphi\right)}\Theta\left(t_1\right). \end{aligned} \tag{50}$$

Since the total reflected pulse $U^{\rm out}(t_1)$ is a superposition of the CAOM with the AOM, i.e.,

$$U^{\rm out}(t_1) = \sqrt{\underline{\eta}_{\rm ref}}U_0^{\rm out}(t_1)+\sqrt{\overline{\eta}_{\rm ref}}U_1^{\rm out}(t_1)\,, \tag{51}$$

the form of the AOM, denoted as $U_1^{\rm out}(t_1)$, can be found as

$$\begin{aligned} U_1^{\rm out}(t_1) &= \frac{1}{\sqrt{\overline{\eta}_{\rm ref}}}\left[U^{\rm out}(t_1)-\sqrt{\underline{\eta}_{\rm ref}}\,U_0^{\rm out}(t_1)\right] \\ &= \sqrt{\Gamma}e^{i\chi}\left(\Gamma t_1-1\right)e^{-\left(i\omega_{\rm cav}+\frac{\Gamma}{2}\right)t_1}\Theta\left(t_1\right). \end{aligned} \tag{52}$$

Here

$$\chi = \arg\frac{\mathcal{T}^{(\rm o)}\mathcal{T}^{(\rm c)}}{\Gamma}+\arg\mathcal{T}^{(\rm o)}-\varphi \tag{53}$$

and

$$\overline{\eta}_{\rm ref} = \frac{\left|\mathcal{T}^{(\rm o)}\right|^2\left|\mathcal{T}^{(\rm c)}\right|^2}{\Gamma^2} \tag{54}$$

is the efficiency of the reflection of the MIM into the AOM, which is found via the normalization of the function $U_0^{\rm out}(t_1)$.

One can check by direct calculations that in the new representation Eq. (41) reads

$$\hat{b}_{{\rm out};0} = \sqrt{\eta_{\rm ext}}\,\hat{a}_{\rm cav}(0)+\sqrt{\underline{\eta}_{\rm ref}}\,\hat{b}_{{\rm in};0}+\hat{C}_0, \tag{55}$$

$$\hat{b}_{{\rm out};1} = \sqrt{\overline{\eta}_{\rm ref}}\,\hat{b}_{{\rm in};0}+\sum_{m=1}^{+\infty}G_{m,1}^{*}\,\hat{b}_{\rm in;m}+\hat{C}_1, \tag{56}$$

$$\hat{b}_{{\rm out};n} = \sum_{m=1}^{+\infty}G_{m,n}^{*}\,\hat{b}_{{\rm in};m}+\hat{C}_n \text{ for } n=2,3,\ldots, \tag{57}$$

where

$$G_{m,n}^{*} = \int_{-\infty}^{+\infty}{\rm d}t_1{\rm d}t_2 U_n^{\rm out*}(t_1)\,G^{*}\left(t_1,t_2\right)U_m^{\rm in}\left(t_2\right), \tag{58}$$

$$\hat{C}_n = \int_{-\infty}^{+\infty}{\rm d}t_1 U_n^{\rm out*}(t_1)\,\hat{C}\left(t_1\right). \tag{59}$$

The first term of Eq. (55) describes the intracavity-field extraction into the CAOM with the efficiency $\eta_{\rm ext}$ [14]. This mode corresponds to the function $U_0^{\rm out}(t_1)$. The second term of Eq. (55) demonstrates the possibility to combine the MIM and the intracavity mode in the CAOM with the efficiency $\underline{\eta}_{\rm ref}$ given by Eq. (48). As it follows from Eqs. (56) and (57), the field extracted from the cavity does not give a contribution to other nonmonochromatic output modes. Moreover, according to Eq. (55), only the MIM described by the function $U_0^{\rm in}(t_1)$ contributes into the CAOM via reflection at the cavity. It is worth noting that the MIM can be easily prepared in an experiment since it has the form of a pulse extracted from another cavity of the same type.

The frequency representation of the CAOM and the AOM have a very similar form. Their Fourier images, denoted as $U_0^{\rm out}(\omega)$ and $U_1^{\rm out}(\omega)$ respectively, have equal absolute values, i.e.,

$$\left|U_0^{\rm out}(\omega)\right|^2 = \left|U_1^{\rm out}(\omega)\right|^2 = \frac{\Gamma}{2\pi\left[\left(\omega-\omega_{\rm cav}\right)^2+\frac{\Gamma^2}{4}\right]}. \tag{60}$$

Hence, these two orthogonal modes are irradiated in the same frequency domain. They differ only in the phases.

For the cavity associated with the degenerate replacement scheme in Fig. 3, the efficiency of the reflection of the MIM into the CAOM, see Eq. (48), is zero due the constraint (37). Therefore, the incomplete model does not describe the possibility of the input and intracavity mode matching. Cavities without channels of unwanted noise will not give rise to the mode matching as well.

V. APPLICATION TO QUANTUM-STATE RECONSTRUCTION

The considered mode-coupling effect can be used for unbalanced [29] and cascaded [30] homodyning of the intracavity mode. Presently known methods for the reconstruction of the quantum state of the intracavity mode are based either on an interaction between atoms and intracavity field [31] or on the balanced homodyning of the extracted field [8]. Including in the model unwanted noise, which exists for all realistic cavities, allows one to

formulate another way for the quantum-state reconstruction of the intracavity mode.

The proposed method has two major advantages. First, unbalanced homodyning allows one to perform a local reconstruction of the quantum state of the intracavity mode. That is, in contrast to balanced homodyning, in unbalanced homodyning it is not required to perform complicated integral transformations of measured data. Second, one can directly use properties of the cavity to combine the signal field with the local oscillator, which allows to avoid losses associated with the additional beam splitter. Both features are important in the considered case, because the quantum-state extraction is typically characterized by a small efficiency [11] that gives additional difficulties in the numerical evaluation of the measured data.

A. Unbalanced homodyning

Let us assume that a quantum state of radiation has been generated inside the cavity. The local-oscillator field with the coherent amplitude β is prepared in the form of the MIM, e.g., it could be extracted from another cavity. The further calculations are similar to those in Ref. [29]. The difference is that the influence of the AOM must be taken into account. The photodetector counts the photon number of the total outgoing field

$$\hat{n}_{\rm out} = \hat{b}^{\dagger}_{{\rm out};0}\,\hat{b}_{{\rm out};0} + \hat{b}^{\dagger}_{{\rm out};1}\,\hat{b}_{{\rm out};1} + \ldots , \tag{61}$$

The probability of recording n counts reads

$$p_n = \left\langle : \frac{(\eta_{\rm c}\hat{n}_{\rm out})^n}{n!} \exp\left(-\eta_{\rm c}\hat{n}_{\rm out}\right) : \right\rangle , \tag{62}$$

where :: means normally ordering and $\eta_{\rm c}$ denotes the counting efficiency. The s-parametrized phase space distribution $P_{\rm cav}(\alpha; s)$ of the intracavity mode is expressed in terms of the Glauber-Sudarshan P distribution $P_{\rm cav}(\alpha)$ in the form [32]

$$P_{\rm cav}(\alpha, s) = \frac{2}{\pi\,(1-s)} \int {\rm d}^2\alpha' \exp\left(-\frac{2\,|\alpha-\alpha'|^2}{1-s}\right) P_{\rm cav}(\alpha') , \tag{63}$$

which can be rewritten as

$$P_{\rm cav}(\alpha, s) = \frac{2}{\pi\,(1-s)} \left\langle : \exp\left(-\frac{2}{1-s}\,\hat{n}_{\rm cav}(\alpha)\right) : \right\rangle , \tag{64}$$

where

$$\hat{n}_{\rm cav}(\alpha) = \left(\hat{a}^{\dagger}_{\rm cav}(0) - \alpha^{*}\right)\left(\hat{a}_{\rm cav}(0) - \alpha\right) \tag{65}$$

is the displaced photon-number operator of the intracavity mode. Utilizing the input-output relations (55), (56) and Eq. (61), and assuming that the MIM is in a coherent state of amplitude β and all other modes are in the vacuum state, the operator $\hat{n}_{\rm cav}(\alpha)$ in Eq. (64) can be written in the form

$$\hat{n}_{\rm cav}(\alpha) = \frac{1}{\eta_{\rm ext}}\,\hat{n}_{\rm out} - \epsilon\,|\alpha|^2 , \tag{66}$$

where

$$\alpha = -\sqrt{\frac{\underline{\eta}_{\rm ref}}{\eta_{\rm ext}}}\,\beta, \tag{67}$$

and the factor $\epsilon = \overline{\eta}_{\rm ref}/\underline{\eta}_{\rm ref}$ can be rewritten with the help of Eqs. (48), (54) as

$$\epsilon = \frac{1}{\left|1 + \frac{\mathcal{R}^{(\rm o)}\Gamma}{\mathcal{T}^{(\rm o)}\mathcal{T}^{(\rm c)}}\right|^2}. \tag{68}$$

The second term in Eq. (66), caused by nonzero ϵ, describes the influence of the AOM.

This gives a possibility to rewrite Eq. (64) in the form

$$P_{\rm cav}(\alpha, s) = \frac{2}{\pi\,(1-s)} \exp\left(\frac{2}{1-s}\epsilon\,|\alpha|^2\right) \times \left\langle : \exp\left(-\frac{2}{(1-s)\,\eta_{\rm ext}}\,\hat{n}_{\rm out}\right) : \right\rangle . \tag{69}$$

Similar to Ref. [29], one can decompose the normally ordered exponent into the factor $\exp\left(-\eta_{\rm c}\hat{n}_{\rm out}\right)$ contained in Eq. (62) and a residual factor,

$$P_{\rm cav}(\alpha, s) = \frac{2}{\pi\,(1-s)} \exp\left(\frac{2}{1-s}\epsilon\,|\alpha|^2\right) \times \langle : \exp\left(-\xi\,\eta_{\rm c}\hat{n}_{\rm out}\right) \exp\left(-\eta_{\rm c}\hat{n}_{\rm out}\right) : \rangle , \tag{70}$$

where

$$\xi = \frac{2-\eta(1-s)}{\eta(1-s)} \tag{71}$$

and η is the overall efficiency of detection

$$\eta = \eta_{\rm ext}\eta_{\rm c}. \tag{72}$$

Expanding the residual factor into a series, it is straightforward to find the s-parametrized phase-space distribution

$$P_{\rm cav}(\alpha; s) = \tfrac{2}{\pi(1-s)} \exp\left(\tfrac{2}{1-s}\epsilon\,|\alpha|^2\right) \times \sum_{n=0}^{+\infty} (-\xi)^n\, p_n(\alpha;\eta,\epsilon) , \tag{73}$$

where $p_n(\alpha;\eta,\epsilon) \equiv p_n$ is the probability of recording n counts given by Eq. (62).

Thus, measuring the photocount statistics of the outgoing field p_n, one can reconstruct the s-parametrized phase-space distribution of the intracavity mode. It is worth noting that such a reconstruction is impossible for

cavities without channels of unwanted noise and cavities whose channels of unwanted noise can be modeled, for example, by a degenerate replacement scheme of the type shown in Fig. 3. From Eq. (67) it follows that for such cavities $\underline{\eta}_{\rm ref} = 0$, hence the reconstruction is only possible for $\alpha = 0$, i.e., for the origin of the phase space. In contrast, if the unwanted noise sources are described properly, the complete information about a quantum state of the intracavity mode can be obtained.

Unlike the case considered in Ref. [29], the reliability of the method depends not only on the value of the parameter ξ, but also on the parameter ϵ, cf. Eqs (71) and (68). For the best convergence of the series in Eq. (73), both these parameters should be less than unity. Nevertheless, it is impossible to satisfy these two conditions simultaneously in realistic situations.

To illustrate the method, we consider a cavity with zero absorption coefficient $\mathcal{A}^{(\rm o)}$ of the input field, i.e., with $\left|\mathcal{R}^{(\rm o)}\right|^2 = 1$. As it follows from the constraints (30-32) the value of parameter ϵ given by Eq. (68) can be written in the form

$$\epsilon = \left(\frac{\eta_{\rm ext}}{1-\eta_{\rm ext}}\right)^2. \tag{74}$$

This is a rising function of $\eta_{\rm ext}$, in contrast to the dependence on $\eta_{\rm ext}$ of ξ, see Fig. 5. The intersection of these curves can be considered as the optimum value of $\eta_{\rm ext}$. Even for an ideal detector with $\eta_{\rm c} = 1$, this value is $\eta_{\rm ext} = 0.5$ and corresponding value of the parameters ξ and ϵ is $\xi = \epsilon = 1$. Let us consider a photodetector

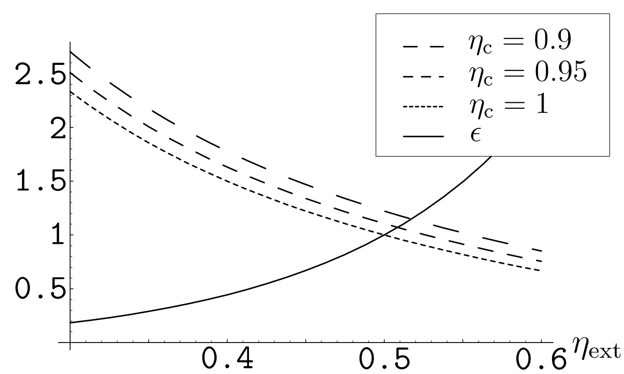

FIG. 5: The dependences of ϵ and ξ on $\eta_{\rm ext}$ for the reconstruction of the Husimi-Kano Q function, $s = -1$, where ξ is given for three different efficiencies of photocounting $\eta_{\rm c}$.

with $\eta_{\rm c} = 0.95$. In this case, the optimum value of the efficiency of quantum-state extraction is $\eta_{\rm ext} = 0.5085$ which corresponds to the situation in Ref. [11]. The corresponding values of the parameters ξ and ϵ for this case are $\xi = \epsilon = 1.070$. In Fig. 6 the result of a numerical simulation of the reconstruction of the Husimi-Kano Q function for the odd superposition of coherent states is presented. Each point is evaluated with 1.7×10^5 sampling events. This is much more than the number of 5×10^3 events needed for such an experiment in the case of usual unbalanced homodyning with the same overall efficiency. In particular, for the phase-space distributions far from the origin of the phase space one needs a large number of sampling events. The method works well for small values of $|\alpha|$, for which $\sim 10^4$ sampling events are sufficient.

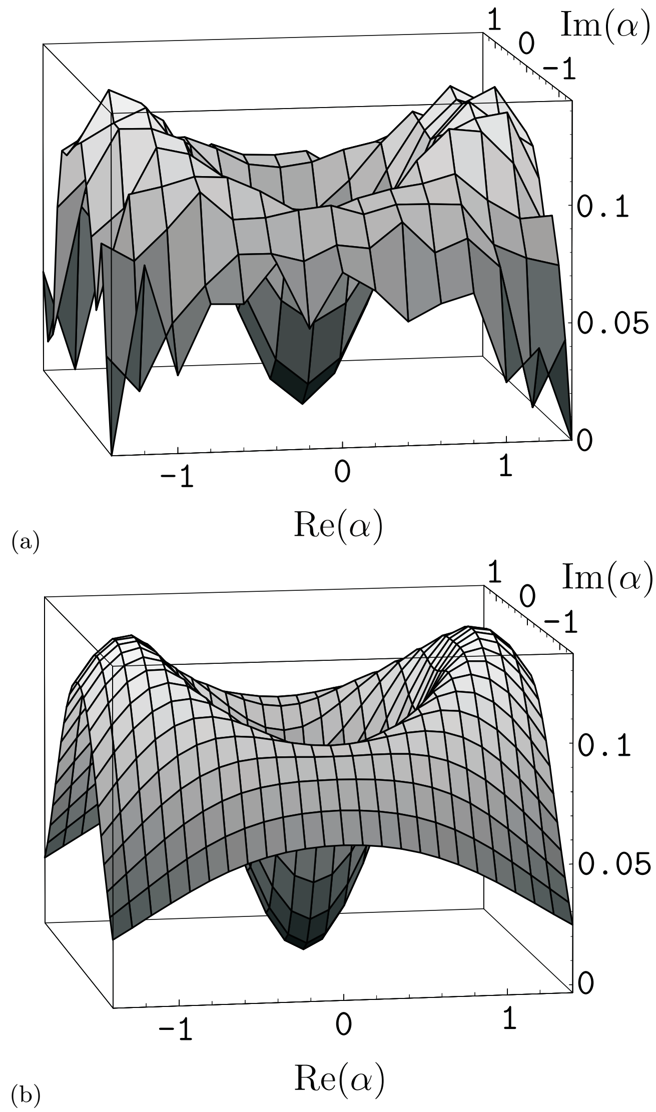

FIG. 6: Reconstruction of the Husimi-Kano Q function for the odd superposition of the coherent states $|\delta_-\rangle = \mathcal{N}\big(|\delta\rangle - |-\delta\rangle\big)$, $\delta = 0.7$, by unbalanced homodyning: (a) numerical simulation with 1.7×10^5 sampling events for each point; (b) exact function.

B. Cascaded homodyning

The efficiency of the scheme can be sufficiently improved by using the related scheme of cascaded homodyning [30]. In this scheme the balanced homodyne detection is used for counting photons [33] in the scheme

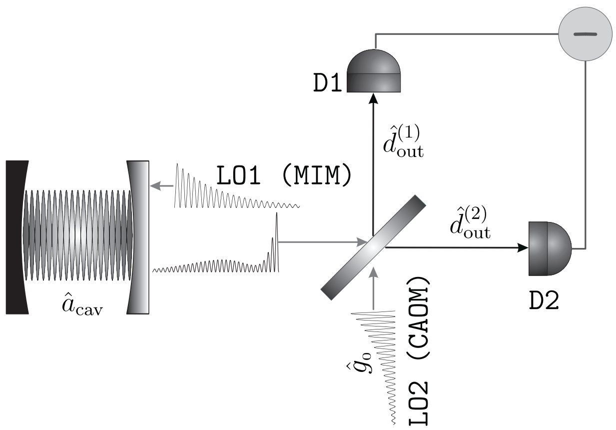

FIG. 7: Cascaded homodyne detection of the intracavity mode.

of unbalanced homodyning, see Fig. 7. The local oscillator 1 (LO1) is prepared in the form of the MIM similar to the case of unbalanced homodyning. The phase randomized local oscillator 2 (LO2) is prepared in the form of the CAOM and it can be derived from the MIM, cf. Eq. (47). In this case the influence of the AOM disappears completely. Hence the results of the work [30] with the overall efficiency given by Eq. (72) can be directly applied to this case.

Let us consider this scheme in more details. The photodetectors D1 and D2 count the photon numbers in the output ports of a 50 : 50 beam splitter,

$$\hat{n}_{\mathrm{out}}^{(k)} = \hat{d}_{\mathrm{out};0}^{\dagger(k)}\,\hat{d}_{\mathrm{out};0}^{(k)} + \hat{d}_{\mathrm{out};1}^{\dagger(k)}\,\hat{d}_{\mathrm{out};1}^{(k)} + \ldots, \tag{75}$$

where $k = 1, 2$ is the number of the detector. The operators $\hat{d}_{\mathrm{out};n}^{(k)}$ are connected with the mode operators $\hat{b}_{\mathrm{out};n}$ of the cavity output and the operators $\hat{g}_{\mathrm{n}}$ describing the LO2 via the relations

$$\hat{d}_{\mathrm{out};n}^{(1)} = \frac{1}{\sqrt{2}}\left(\hat{b}_{\mathrm{out};n} + \hat{g}_n\right), \tag{76}$$

$$\hat{d}_{\mathrm{out};n}^{(2)} = \frac{1}{\sqrt{2}}\left(-\hat{b}_{\mathrm{out};n} + \hat{g}_n\right). \tag{77}$$

The LO2 corresponds to the MIM in a phase-randomized coherent state, i.e.,

$$\hat{g}_0 \to re^{i\varphi}, \tag{78}$$

where r is the amplitude and φ is the random phase. Whereas the AOM is superposed with the corresponding mode of LO2 being in the vacuum state

$$\hat{g}_1 \to 0. \tag{79}$$

The difference of the photocounts $\hat{n}_{\mathrm{out}}^{(1)}$ and $\hat{n}_{\mathrm{out}}^{(2)}$ can be written in the form

$$\hat{n}_{\mathrm{out}}^{(1)} - \hat{n}_{\mathrm{out}}^{(2)} = r\sqrt{2}\hat{x}(\varphi), \tag{80}$$

where

$$\hat{x}(\varphi) = \frac{1}{\sqrt{2}}\left(\hat{b}_{\mathrm{out};0}^{\dagger}e^{i\varphi} + \hat{b}_{\mathrm{out};0}e^{-i\varphi}\right) \tag{81}$$

is the quadrature operator of the CAOM. This means that the AOM does not affect in cascaded homodyning.

Let $p(x;\alpha,\eta)$ be the phase-averedged quadrature distribution measured with the shifted amplitude α and the efficiency η given by Eqs. (67) and (72), respectively. Utilizing the results of Refs. [30, 34], one can write the s-parametrized phase-spase distribution of the intracavity mode as

$$P_{\mathrm{cav}}(\alpha;s) = \int_{-\infty}^{+\infty} \mathrm{d}x\, S\,(x;s,\eta)\, p(x;\alpha,\eta), \tag{82}$$

where the sampling function $S\,(x;s,\eta)$ has the form

$$S\,(x;s,\eta) = \frac{\eta}{\pi\left[\eta(1-s)-1\right]}\, f_{00}\left(\frac{x}{\sqrt{\eta(1-s)-1}}\right), \tag{83}$$

with $f_{00}(x)$ being expressed in terms of the Dawson integral $F(x) = e^{-x^2}\int_0^x \mathrm{d}t\, e^{t^2}$ as

$$f_{00}(x) = 2 - 4xF(x). \tag{84}$$

One can use Eq. (82) for the reconstruction of the phase-space distribution of the intracavity mode. It is worth noting that the reliability of this method is completely the same as in the case of a free signal field considered in Ref. [30], since the influence of the AOM is eliminated by the technique itself.

VI. SUMMARY AND CONCLUSIONS

For high-Q cavities, unwanted losses such as the losses due to scattering and/or absorption may be of the same order of magnitude as the wanted losses due to the fractional transparency of the coupling mirrors. When such cavities are used for the generation and transfer of nonclassical light, it is of great importance to carefully consider the noise effects caused by all the unwanted dissipative channels.

In the present paper we have derived a rather simple and intuitive extension of the standard quantum noise theory in order to include in the theory unwanted losses in a consistent way. For this purpose, we have modeled the cavity losses by additional beam splitters that are placed in the input and output channels of the radiation. We have analyzed the requirements and constraints for a complete description of the unwanted losses. Most importantly, such a model must ensure that the fundamental commutation rules remain valid, which allows one to study the possibilities of a complete parametrization of a cavity with unwanted noise.

To illustrate the relevance of a correct and complete parameterization, we have also considered an example

of a degenerate model. It shows that, even though unwanted dissipative channels are included in the model, the situation may resemble that of a cavity without unwanted noise. For such cavities information about the relative phase between intracavity and input modes does not exist in the outgoing field. In fact, combining the intracavity mode and the input mode in the nonmonochromatic output mode becomes possible due to the presence of proper losses inside the cavity and the coupling mirror.

This mode matching effect can be used, for example, for homodyne measurements of the intracavity mode. Due to the presence of the additional output mode satisfactory results for large amplitudes can be obtained only with a large number of sample events. However, by using cascaded homodyning the influence of the additional output mode play no role anymore. In this case the applicability of the method is much better. The proposed scheme has two advantages compared with standard homodyning of the extracted field. First, the phase-space distribution is reconstructed in a point, which is specified by the value of the local-oscillator amplitude. This implies that one avoids a complicated numerical integration of experimental data. Second, one may directly use the properties of the cavity for combining the signal field with the local oscillator, avoiding additional mode matching by a beam-splitter and the related losses. Due to the small efficiency of the quantum-state extraction from a high-Q cavity, these properties can be useful for increasing the overall efficiency of the scheme and, consequently, for obtaining more detailed information about the quantum state of the intracavity mode.

Acknowledgments

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APPENDIX A: SYMMETRICAL AND ASYMMETRICAL BEAM SPLITTERS

Let us briefly explain some features of the input-output relation for the two types of beam-splitters which appear in the replacement schemes: symmetrical and asymmetrical beam splitters. A symmetrical beam splitter is a four-port device that is described by the SU(2) group. The corresponding input-output relations can be written as

$$\hat{a}_{\rm out}^{(k)} = \mathcal{T}^{(k)}\hat{a}_{\rm in}^{(k)} + \mathcal{R}^{(k)}\hat{b}_{\rm in}^{(k)}, \tag{A1}$$

$$\hat{b}_{\rm out}^{(k)} = -\mathcal{R}^{(k)*}\hat{a}_{\rm in}^{(k)} + \mathcal{T}^{(k)*}\hat{b}_{\rm in}^{(k)}. \tag{A2}$$

Inverting these equations, we arrive at

$$\hat{a}_{\rm in}^{(k)} = \mathcal{T}^{(k)*}\hat{a}_{\rm out}^{(k)} - \mathcal{R}^{(k)}\hat{b}_{\rm out}^{(k)}, \tag{A3}$$

$$\hat{b}_{\rm in}^{(k)} = \mathcal{R}^{(k)*}\hat{a}_{\rm out}^{(k)} + \mathcal{T}^{(k)}\hat{b}_{\rm out}^{(k)}. \tag{A4}$$

Here, the index k refers to the respective beam splitter. For example, for the beam splitter BS_1 in the replacement scheme in Fig. 2, we have $\hat{a}_{\rm in}^{(1)} = \hat{g}_{\rm in}$, $\hat{b}_{\rm in}^{(1)} = \hat{c}_{\rm in}^{(1)}$, $\hat{a}_{\rm out}^{(1)} = \hat{d}_{\rm in}$, and $\hat{b}_{\rm out}^{(1)} = \hat{c}_{\rm out}^{(1)}$. The transmission and reflection coefficients $\mathcal{T}^{(k)}$ and $\mathcal{R}^{(k)}$, respectively, which satisfy the condition

$$\left|\mathcal{T}^{(k)}\right|^2 + \left|\mathcal{R}^{(k)}\right|^2 = 1 \tag{A5}$$

can be parametrized by three real numbers $\theta^{(k)}$, $\mu^{(k)}$, and $\nu^{(k)}$ in the form

$$\mathcal{T}^{(k)} = \cos\theta^{(k)} e^{i\mu^{(k)}}, \tag{A6}$$

$$\mathcal{R}^{(k)} = \sin\theta^{(k)} e^{i\nu^{(k)}}. \tag{A7}$$

It is clear that the determinant of the transform matrix is equal to 1 in this case—the case of a symmetrical beam splitter.

In the case of an asymmetrical beam splitter, the determinant is an arbitrary phase multiplier. In fact, this means that this multiplier should be included in the input-output relations, which then read

$$\hat{a}_{\rm out}^{(k)} = e^{i\varphi^{(k)}}\mathcal{T}^{(k)}\hat{a}_{\rm in}^{(k)} + e^{i\varphi^{(k)}}\mathcal{R}^{(k)}\hat{b}_{\rm in}^{(k)}, \tag{A8}$$

$$\hat{b}_{\rm out}^{(k)} = -\mathcal{R}^{(k)*}\hat{a}_{\rm in}^{(k)} + \mathcal{T}^{(k)*}\hat{b}_{\rm in}^{(k)}. \tag{A9}$$

This is also a unitary transformation, however a U(2)-group transformation. Inverting Eqs. (A8) and (A9) yields

$$\hat{a}_{\rm in}^{(k)} = e^{-i\varphi^{(k)}}\mathcal{T}^{(k)*}\hat{a}_{\rm out}^{(k)} - \mathcal{R}^{(k)}\hat{b}_{\rm out}^{(k)}, \tag{A10}$$

$$\hat{b}_{\rm in}^{(k)} = e^{-i\varphi^{(k)}}\mathcal{R}^{(k)*}\hat{a}_{\rm out}^{(k)} + \mathcal{T}^{(k)}\hat{b}_{\rm out}^{(k)}. \tag{A11}$$

The quantities $\mathcal{T}^{(k)}$ and $\mathcal{R}^{(k)}$ again satisfy the condition (A5) and can be parametrized according to Eqs. (A6) and (A7). Clearly the transformation matrix depends on the additional parameter $\varphi^{(k)}$ and hence, the resulting number of independent parameters, describing an asymmetrical beam-splitter is equal to 4.

APPENDIX B: QUANTUM LANGEVIN EQUATION AND INPUT-OUTPUT RELATION

Let us start with the derivation of the quantum Langevin equation (3). Utilizing the input-output relation (2) as well as the input-output relations for each beam-splitter in Fig. 2 (see Appendix A), we have to first express the operator $\hat{d}_{\rm in}(t)$ in terms of the operators $\hat{b}_{\rm in}(t)$, $\hat{c}_{\rm in}^{(1)}(t)$, $\hat{c}_{\rm in}^{(2)}(t)$, and $\hat{a}_{\rm cav}(t)$. For this purpose we apply the input-output relation for the first beam splitter

$$\hat{d}_{\rm in}(t) = \mathcal{T}^{(1)}\hat{g}_{\rm in}(t) + \mathcal{R}^{(1)}\hat{c}_{\rm in}^{(1)}(t), \tag{B1}$$

and then we find an appropriate expression for the operator $\hat{g}_{\rm in}(t)$. Further, we apply the input-output relations for the beam splitters in Fig. 2 starting from the third one and moving clockwise in the loop. The formal sequence of operations looks as follows.

1. Substitute $\hat{g}_{\rm out}(t)$ from the input-output relation

$$\hat{g}_{\rm out}(t) = \mathcal{T}^{(2)}\hat{d}_{\rm out}(t) + \mathcal{R}^{(2)}\hat{c}^{(2)}_{\rm in}(t) \tag{B2}$$

for the second beam splitter into the input-output relation

$$\hat{g}_{\rm in}(t) = -\mathcal{R}^{(3)*}\hat{g}_{\rm out}(t) + \mathcal{T}^{(3)*}\hat{b}_{\rm in}(t) \tag{B3}$$

for the third beam splitter.

2. Substitute in the resulting equation $\hat{d}_{\rm out}(t)$ from the input-output relations for the cavity, Eq. (2).

3. Substitute in the resulting equation $\hat{d}_{\rm in}(t)$ from Eq. (B1). This leads to

$$\begin{aligned}\hat{g}_{\rm in}(t) = &-\mathcal{R}^{(3)*}\mathcal{T}^{(2)}\sqrt{\gamma}\,\hat{a}_{\rm cav}(t) + \mathcal{T}^{(3)*}\hat{b}_{\rm in}(t)\\ &+\mathcal{T}^{(2)}\mathcal{R}^{(1)}\mathcal{R}^{(3)*}\hat{c}^{(1)}_{\rm in}(t) - \mathcal{R}^{(2)}\mathcal{R}^{(3)*}\hat{c}^{(2)}_{\rm in}(t)\\ &+\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}\hat{g}_{\rm in}(t),\end{aligned} \tag{B4}$$

which contains the operator $\hat{g}_{\rm in}(t)$ in both sides of the equation.

4. Resolve Eq. (B4) to find $\hat{g}_{\rm in}(t)$,

$$\begin{aligned}\hat{g}_{\rm in}(t) = &-\sqrt{\gamma}\,\frac{\mathcal{R}^{(3)*}\mathcal{T}^{(2)}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,\hat{a}_{\rm cav}(t) + \frac{\mathcal{T}^{(3)*}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,\hat{b}_{\rm in}(t)\\ &+\frac{\mathcal{T}^{(2)}\mathcal{R}^{(1)}\mathcal{R}^{(3)*}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,\hat{c}^{(1)}_{\rm in}(t) - \frac{\mathcal{R}^{(2)}\mathcal{R}^{(3)*}}{1-\mathcal{R}^{(3)*}\mathcal{T}^{(1)}\mathcal{T}^{(2)}}\,\hat{c}^{(2)}_{\rm in}(t).\end{aligned} \tag{B5}$$

Inserting Eq. (B5) into Eq. (B1) and then substituting the result into Eq. (1), we obtain the quantum Langevin equation (3). Next, combining Eq. (B5) with the (inverse) input-output relation

$$\hat{b}_{\rm in}(t) = e^{-i\varphi^{(3)}}\mathcal{R}^{(3)*}\hat{b}_{\rm out}(t) + \mathcal{T}^{(3)}\hat{g}_{\rm in}(t) \tag{B6}$$

for the third (asymmetrical) beam splitter in Fig. 2, we arrive at the sought input-output relation (4).

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